6/10/22

<u>Recall Def</u>: An orientation N of M is a smooth unit normal vector field:

- N is smooth
- N has unit length
- $\forall p$, $N \perp T_pM$.

$$N = \frac{X_u \times X_v}{|X_u \times X_v|}$$

The shape operator / Weingarten map w.r.t. N at p is defined as the following: let $v \in T_pM$, $\alpha(t)$ smooth curve, $\alpha(0) = p$, $\alpha'(0) = v$, then

$$S_p(v) = -\frac{d}{dt}\left(N(\alpha(t))\right)\Big|_{t=0}.$$

$\Leftarrow$ differential of "Gauss map" in do Carmo.

$S_p(X_u) = -N_u$, $S_p(X) = -N_v$.

The second fundamental form $\mathrm{II}_p : T_pM \times T_pM \to \mathbb{R}$.

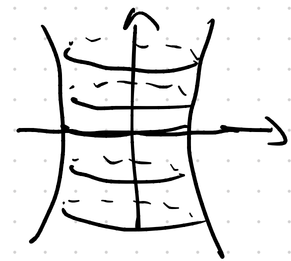
$$\mathrm{II}_p(v,w) = g\left(S_p(v), w\right) = \langle S_p(v), w \rangle.$$

$$e := \mathrm{II}_p(X_u, X_u) = -\langle N_u, X_u \rangle$$
$$f := \mathrm{II}_p(X_u, X_v) = -\langle N_u, X_v \rangle$$
$$g := \mathrm{II}_p(X_v, X_v) = -\langle N_v, X_v \rangle.$$

$\rightsquigarrow$ notion of Gauss Curvature $K(p)$
Mean Curvature $H(p)$.

Ex: Given parametrization of the catenoid

$$X(u,v) = \left(c\cosh\left(\tfrac{v}{c}\right)\cos(u),\ c\cosh\left(\tfrac{v}{c}\right)\sin(u),\ v\right) \quad c>0$$



Compute N, $S_p(X_u)$, $S_p(X_v)$.

$$X_u = \left(-c\cosh\left(\tfrac{v}{c}\right)\sin(u),\ c\cosh\left(\tfrac{v}{c}\right)\cos(u),\ 0\right)$$

$$X_v = \left(\sinh\left(\tfrac{v}{c}\right)\cos(u),\ \sinh\left(\tfrac{v}{c}\right)\sin(u),\ 1\right)$$

$$X_u \times X_v = \left(c\cosh\left(\tfrac{v}{c}\right)\cos(u),\ c\cosh\left(\tfrac{v}{c}\right)\sin(u),\ -c\cosh\left(\tfrac{v}{c}\right)\sinh\left(\tfrac{v}{c}\right)\right)$$

$$|X_u \times X_v| = \left(c^2\cosh^2\left(\tfrac{v}{c}\right) + c^2\cosh^2\left(\tfrac{v}{c}\right)\sinh^2\left(\tfrac{v}{c}\right)\right)^{1/2} = c\cosh^2\left(\tfrac{v}{c}\right). \qquad (1+\sinh^2 = \cosh^2)$$

$$N = \left(\frac{\cos(u)}{\cosh\left(\frac{v}{c}\right)},\ \frac{\sin(u)}{\cosh\left(\frac{v}{c}\right)},\ -\tanh\left(\tfrac{v}{c}\right)\right).$$

$$S_p(X_u) = -N_u = \left(\frac{\sin(u)}{\cosh\left(\frac{v}{c}\right)}, \frac{-\cos(u)}{\cosh\left(\frac{v}{c}\right)}, 0\right)$$

$$S_p(X_v) = -N_v = \frac{1}{c}\left(\cos(u)\tanh\left(\tfrac{v}{c}\right)\operatorname{sech}\left(\tfrac{v}{c}\right), \sin(u)\tanh\left(\tfrac{v}{c}\right)\operatorname{sech}\left(\tfrac{v}{c}\right), \operatorname{sech}^2\left(\tfrac{v}{c}\right)\right).$$

<u>Ex</u>: Compute e, f, g

$$e = -\langle N_u, X_u\rangle = -c\sin^2(u) - c\cos^2(u) = -c$$

$$f = -\langle N_u, X_v\rangle = \frac{\sin(u)\cos(u)\sinh\left(\frac{v}{c}\right)}{\cosh\left(\frac{v}{c}\right)} - \frac{\sin(u)\cos(u)\sinh\left(\frac{v}{c}\right)}{\cosh\left(\frac{v}{c}\right)} = 0$$

$$g = -\langle N_v, X_v\rangle = \frac{1}{c}\left(\cos^2(u)\tanh^2\left(\tfrac{v}{c}\right) + \sin^2(u)\tanh^2\left(\tfrac{v}{c}\right) + \operatorname{sech}^2\left(\tfrac{v}{c}\right)\right)$$

$$= \frac{1}{c}\left(\tanh^2\left(\tfrac{v}{c}\right) + \operatorname{sech}^2\left(\tfrac{v}{c}\right)\right) = \frac{1}{c}.$$

$$H(p) = \frac{1}{2}\,\frac{eG - 2fF + gE}{EG - F^2}, \quad E = c^2\cosh^2\left(\tfrac{v}{c}\right), \; G = \cosh^2\left(\tfrac{v}{c}\right), \; F = 0$$

$$= \frac{1}{2}\,\frac{\cancel{(-c)\cosh^2\left(\frac{v}{c}\right) + \frac{1}{c}c^2\cosh^2\left(\frac{v}{c}\right)}}{c^2\cosh^4\left(\frac{v}{c}\right)} = 0.$$

So this verifies that the catenoid is a minimal surface.